

CRITICAL MANDELBROT CASCADES

JULIEN BARRAL, ANTTI KUPIAINEN^{1,2}, MIIKA NIKULA¹, EERO SAKSMAN¹,
AND CHRISTIAN WEBB¹

ABSTRACT. We study Mandelbrot’s multiplicative cascade measures at the critical temperature. As has been recently shown by Barral, Rhodes and Vargas ([11]), an appropriately normalized sequence of cascade measures converges weakly in probability to a nontrivial limit measure. We prove that these limit measures have no atoms and give bounds for the modulus of continuity of the cumulative distribution function of the measure. Using the earlier work of Barral and Seuret ([12]), we compute the multifractal spectrum of the measures. We also extend the result of Benjamini and Schramm ([13]), in which the KPZ formula from quantum gravity is validated for the high temperature cascade measures, to the critical and low temperature cases.

1. INTRODUCTION

Random multiplicative cascade measures were introduced by B. Mandelbrot [38],[39],[37], as simple models exhibiting fractal and statistical features analogous to those observed experimentally in velocity fluctuations of fully developed turbulence. Since then these multifractal measures have found applications in various fields ranging from mathematical finance to disordered systems and two dimensional quantum gravity (see [7] for references). In the field of disordered systems the (normalized) cascade measures can be seen as Gibbs measures of Generalized Random Energy Models with infinitely many levels or continuous hierarchies (see e.g. [17, 18]) or alternatively as Gibbs measures of a model of a directed polymer on a disordered tree [23]. The mathematical study of multiplicative cascades was initiated by Kahane [32] and Peyrière [41] and has since then been pursued by numerous people in analysis, probability and mathematical physics, often independently (see [9],[10],[5],[44] for references to some of the work).

The simplest cascade measures are random measures on the unit interval defined in terms of two inputs (see below): a real valued random variable ξ (describing fluctuations at a fixed scale) and (inverse) temperature parameter $\beta > 0$. The behavior of the measures is rather insensitive to ξ but depends strongly on β . Derrida and Spohn [23] argued in 1988 that they exhibit a phase transition at a critical value β_c of β to a “glassy” low temperature phase in $\beta > \beta_c$. In the high temperature region $\beta < \beta_c$ the measures

Date: April 30, 2013.

2010 *Mathematics Subject Classification.* Primary 60G57, 28A78; Secondary 60G18, 83C45, 60G51.

Key words and phrases. Multiplicative cascades, critical temperature, KPZ formula, multifractal analysis.

¹Supported by the ¹Academy of Finland and ²ERC

are continuous (but singular with respect to the Lebesgue measure), already proven by Kahane and Peyrière [33] in 1976. Progress in the critical $\beta = \beta_c$ case and the supercritical $\beta > \beta_c$ cases has been slower to come. The reason is that whereas in the subcritical case the cascade measures can be proven to exist as non-degenerate limits of positive martingales, in the critical and supercritical cases the martingale limit vanishes, and there is no obvious candidate for a normalization leading to convergence in law to a non trivial limit. Very recently, Aidékon and Shi [2] proved detailed asymptotics for the probability distribution of the total mass of the cascade measures in the critical case. In the case where ξ is Gaussian the fifth author [44] obtained independently similar results, both in the critical and in the supercritical case, basing his approach on the seminal paper by Bramson [19]. Madaule [36] treated the supercritical case for general ξ . These results allow one to find the required renormalizations and to construct the limits for the total mass (partition function). Recently, this was extended to the measures themselves by Barral, Rhodes and Vargas in [11]. These latter authors also proved that the cascade measures are a.s. purely atomic in the supercritical case.

In this paper we study the cascade measures at the critical point. We give a simple proof that they have a.s. no atoms based on a recent result by Buraczewski in [20] and the aforementioned results on the renormalization factors. We also give bounds for the modulus of continuity of the cumulative distribution function of the critical measure which is of interest for the attempts to use these measures as inputs for construction of random plane curves by conformal welding (see [3], [43] for such constructions in the high temperature case when the cascade measure is replaced by exponential of the Gaussian Free Field, related to a continuous cascade model). In passing we note that our approach can also be used to improve the known bounds for the modulus of continuity of the Mandelbrot measures in the subcritical case. Next, we discuss the KPZ formula [34] of two dimensional quantum gravity in the cascade context. The KPZ formula was reformulated by Duplantier and Sheffield [24] as a relation between Hausdorff dimensions of sets computed with the Lebesgue measure and a random measure given by exponential of the Gaussian Free Field and proven by them to be valid in the high temperature region. In the cascade context the high temperature result was proved by Benjamini and Schramm ([13]) and we show how their proof generalizes to the critical and low temperature cases. Finally, using the earlier work of Barral and Seuret ([12]), we compute the multifractal spectrum of the measures in the critical and low temperature cases.

It remains a challenge to extend the results of this paper and [11] to the stationary log-normal multiplicative chaos of Mandelbrot [37] and the related measures given as exponentials of the Gaussian Free Field (GFF) [24]. Some progress to this goal has been obtained very recently [25],[26],[8]. Especially, in [25] it is shown that in the critical case for GFF there are no atoms, but their methods do not give as fine control of the continuity of the measures as ours. Moreover, [8] establishes partial counterparts of certain results of the present paper for critical GFF.

2. DEFINITIONS AND RESULTS

For simplicity, in this paper we will consider only multiplicative cascade measures on binary trees. We define the symbolic space as $\Sigma = \bigcup_{n=1}^{\infty} \{0, 1\}^n$ and for convenience denote the n -th level by $\Sigma_n = \{0, 1\}^n$ i.e. this set indexes the edges of the tree on n -th level. Let ξ be a random variable such that

$$(1) \quad \mathbb{E}e^{\xi} = \frac{1}{2} \quad \text{and} \quad \mathbb{E}\xi e^{\xi} = 0$$

and

$$(2) \quad \mathbb{E}e^{(1+h)\xi} < \infty \quad \text{for some } h > 0.$$

The conditions (1) are essentially a normalization that is convenient for studying the critical case ($\beta_c = 1$ with this normalization) and can be changed by considering instead $a\xi + b$ for $a, b \in \mathbb{R}$. E.g. in the Gaussian case $\xi \sim N(-2\log 2, 2\log 2)$ satisfies (1).

The condition (2) on the tail behavior of ξ is a technical assumption that is required for the proofs of many of the results we are building upon. It is obviously satisfied in the Gaussian case. Many of our results remain true on less stringent assumptions, and in some cases we indicate this explicitly.

To define the cascade measures, let $\{\xi_{\sigma}\}_{\sigma \in \Sigma}$ be an independent family of copies of ξ and associate to every $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \Sigma$ the sum

$$X_{\sigma} = \xi_{\sigma_1} + \xi_{\sigma_1\sigma_2} + \dots + \xi_{\sigma_1\sigma_2\dots\sigma_n}.$$

For any $\beta > 0$, consider the partition function

$$(3) \quad Z_{\beta,n} = \sum_{\sigma \in \Sigma_n} e^{\beta X_{\sigma}} \quad \text{for } n = 1, 2, \dots$$

In other words, we consider a basic model of the branching random walk with X_{σ} the positions of the 2^n particles at time n . Interpreting $\sigma \in \Sigma_n$ as a spin configuration on $\{1, \dots, n\}$ we recognize that $Z_{\beta,n}$ is the partition function of a Generalized Random Energy Model with continuous hierarchies, see the end of this Section for further discussion. Finally, one may also view $Z_{\beta,n}$ as the partition function of a model for a polymer on a tree [23].

It is a classical result of Kahane and Peyrière [33] that for $\beta < 1$ (the *subcritical* or *high temperature* case) we have

$$(4) \quad (\mathbb{E}Z_{\beta,n})^{-1} Z_{\beta,n} \xrightarrow{n \rightarrow \infty} Y_{\beta} \quad \text{almost surely,}$$

where the limit variable Y_{β} is almost surely positive. It has recently been shown by Aïdékon and Shi ([2]) that for $\beta = 1$ (the *critical case*),

$$(5) \quad n^{\frac{1}{2}} Z_{1,n} \xrightarrow{n \rightarrow \infty} Y_1 \quad \text{in probability,}$$

where Y_1 is an almost surely positive random variable of infinite mean. Another recent result, due to Madaule ([36]), shows that for $\beta > 1$ (the *supercritical* or *low temperature* case) we have

$$(6) \quad n^{\frac{3\beta}{2}} Z_{\beta,n} \xrightarrow{n \rightarrow \infty} Y_{\beta} \quad \text{in distribution}$$

for a positive random variable Y_{β} . In the case of a Gaussian ξ similar results on critical and supercritical cases were obtained independently by the fifth author ([44]) who proved convergence in distribution. It is known that

convergence in (5) (resp. (6)) cannot be improved to almost sure convergence (resp. convergence in probability).

In accordance with these deterministic normalizations for $Z_{\beta,n}$ we study the measures $\mu_{\beta,n}$ on $[0, 1]$ defined by

$$\begin{aligned}\mu_{\beta,n}(I_\sigma) &= (\mathbb{E}Z_{\beta,n})^{-1} e^{\beta X_\sigma} \quad \text{for } \beta < 1, \\ \mu_{1,n}(I_\sigma) &= n^{\frac{1}{2}} e^{X_\sigma}, \quad \text{and} \\ \mu_{\beta,n}(I_\sigma) &= n^{\frac{3\beta}{2}} e^{\beta X_\sigma} \quad \text{for } \beta > 1,\end{aligned}$$

where I_σ is the dyadic interval naturally coded by $\sigma \in \Sigma_n$, and the density of $\mu_{\beta,n}$ with respect to the Lebesgue measure is constant on each of these level n intervals. As said in the introduction, the corresponding limit measures μ_β in the subcritical case have been much studied and well understood, and it holds that

$$(7) \quad \mu_{\beta,n} \xrightarrow[n \rightarrow \infty]{w} \mu_\beta \quad \text{almost surely} \quad \text{for } \beta < 1,$$

where the law of the limit measure satisfies

$$(8) \quad (\mu_\beta(I_\sigma))_{\sigma \in \Sigma_n} \stackrel{d}{=} \left((\mathbb{E}Z_{\beta,n})^{-1} e^{\beta X_\sigma} Y_\beta^{(\sigma)} \right)_{\sigma \in \Sigma_n} \quad \text{for all } n \geq 1,$$

where $\{Y_\beta^{(\sigma)}\}_{\sigma \in \Sigma_n}$ is an independent collection of copies of Y_β that is also independent of $\{X_\sigma\}_{\sigma \in \Sigma_n}$. If for $s \in \mathbb{R}$ we set

$$(9) \quad \phi(s) = -\log_2 \mathbb{E}(e^{s\xi}) \quad \text{and} \quad \tilde{\phi}(s) = 1 - \phi(s)$$

we see that $\mathbb{E}Z_{n,\beta} = 2^{n\tilde{\phi}(\beta)}$.

In the critical and supercritical cases it was observed by Barral, Rhodes and Vargas in [11] that the deterministic normalizations for the partition functions imply the weak convergence of the measures $\mu_{\beta,n}$ to nontrivial limit measures:

$$(10) \quad \mu_{1,n} \xrightarrow[n \rightarrow \infty]{w} \mu_1 \quad \text{in probability} \quad \text{and}$$

$$(11) \quad \mu_{\beta,n} \xrightarrow[n \rightarrow \infty]{w} \mu_\beta \quad \text{in distribution} \quad \text{for } \beta > 1,$$

where the laws of the measures μ_β for $\beta \geq 1$ can be described by

$$(12) \quad (\mu_\beta(I_\sigma))_{\sigma \in \Sigma_n} \stackrel{d}{=} \left(e^{\beta X_\sigma} Y_\beta^{(\sigma)} \right)_{\sigma \in \Sigma_n}, \quad \text{for all } n \geq 1,$$

where $\{Y_\beta^{(\sigma)}\}_{\sigma \in \Sigma_n}$ is an independent collection of copies of Y_β that is also independent of $\{X_\sigma\}_{\sigma \in \Sigma_n}$.

Barral, Rhodes and Vargas noted furthermore that the law of the supercritical limit measures μ_β , $\beta > 1$, may also be described in terms of the law of the critical case μ_1 as follows. For $\alpha \in (0, 1)$, let $(L_\alpha(s))_{s \geq 0}$ be a stable Lévy subordinator of index α , that is, a process with $L_\alpha(0) = 0$ that has independent and stationary increments that are characterized by the Laplace transform

$$\mathbb{E}e^{-uL_\alpha(s)} = e^{-su^\alpha}.$$

Then, if $\mathbb{E}e^{(\beta+\epsilon)X} < \infty$ for some $\epsilon > 0$,

$$(13) \quad \left(\mu_\beta([0, t]) \right)_{t \in [0, 1]} \stackrel{d}{=} \left(cL_{\frac{1}{\beta}}(\mu_1([0, t])) \right)_{t \in [0, 1]}$$

where $L_{\frac{1}{\beta}}$ is taken independent of the critical case measure μ_1 , and $c > 0$ is a constant that depends only on β . Since the process $(L_{\frac{1}{\beta}}(s))_{s \geq 0}$ is a pure jump process, this implies that the measures μ_β , $\beta > 1$, are almost surely purely atomic. For this reason the phase transition in the cascade model at $\beta = 1$ is called in the physics literature *freezing* transition.

It thus remains to understand the critical measure μ_1 which is the topic of the present paper. Our first result shows that μ_1 has no atoms:

Theorem 1. *For any $\gamma \in [0, 1/2)$ we have*

$$(14) \quad n^\gamma \max_{\sigma \in \{0,1\}^n} \mu_1(I_\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty,$$

and for any $\gamma \in (1/2, \infty)$ we have

$$(15) \quad n^\gamma \max_{\sigma \in \{0,1\}^n} \mu_1(I_\sigma) \xrightarrow{\mathbb{P}} \infty \quad \text{as } n \rightarrow \infty.$$

The proof of this result, given in Section 3, uses only elementary tools in combination with two ingredients: the exact asymptotics of the tail of the random variable Y_1 (Theorem 8 below, due to Buraczewski in [20]), and the knowledge of the normalization considered above that is included in the very definition of Y_1 (Theorem 7 below).

Corollary 2. *Almost surely the limit measure μ_1 has no atoms.*

The proof of Theorem 1, in combination with suitable moment estimates (see (26) below) in fact gives us a stronger result, an estimate for the modulus of continuity of the cumulative distribution function of μ_1 .

Theorem 3. *Assume that ξ satisfies in addition to (2) also $\mathbb{E}e^{-h\xi} < \infty$ for some $h > 0$. Then for any $\gamma \in (0, 1/2)$*

$$(16) \quad \mu_1(I) \leq C(\omega) \left(\log \left(1 + \frac{1}{|I|} \right) \right)^{-\gamma}$$

for all subintervals $I \subset [0, 1]$. Here $C(\omega)$ is a random constant, finite almost surely. Moreover, one cannot take $\gamma > 1/2$ in the above statement.

In the subcritical case $\beta \in (0, 1)$, it is known that the measure μ_β has a Hölder modulus of continuity. Indeed, the multifractal formalism (see Section 6 and e.g. [4]) tells you via Legendre transform that uniform Hölder continuity holds with any exponent $\gamma < \tilde{\phi}(\beta)$ and cannot hold for any $\gamma > \tilde{\phi}(\beta)$. It turns out that our proof of Theorem 3 can also be applied to considerably sharpen this result, which up to now has been the best modulus of continuity estimate in the subcritical case.

Theorem 4. *Assume only that $\mathbb{E}|\xi|^3 e^\xi < \infty$. Let $\beta \in (0, 1)$ and $\gamma \in (0, 1/2)$. Suppose that there exists $q_\beta > 1$ such that $\tilde{\phi}(\beta q_\beta) - q_\beta \tilde{\phi}(\beta) = 0$ (in that case q_β is unique and the condition amounts to saying that for $q > 0$ one has $\mathbb{E}Y_\beta^q < \infty$ if and only if $q < q_\beta$). Suppose also that $\mathbb{E} \max(0, \xi) e^{\beta q_\beta \xi} < \infty$. Then for all subintervals $I \subset [0, 1]$*

$$(17) \quad \mu_\beta(I) \leq C(\omega) |I|^{\tilde{\phi}(\beta)} \left(\log \left(1 + \frac{1}{|I|} \right) \right)^{-\gamma \beta}$$

where $C(\omega) < \infty$ almost surely. In the Gaussian case, $\tilde{\phi}(s) = (1 - \beta)^2$ and $q_\beta = 1/\beta^2$.

In the Gaussian case there is another, more sophisticated way to get hold on the size of $\max_{\sigma \in \{0,1\}^n} \mu_n(I_\sigma)$ that is based on the following result.

Theorem 5. *Let ξ be Gaussian and assume that $\beta > 1$. Then there is a deterministic bounded sequence $c(n)$, bounded away from 0 such that*

$$(18) \quad c(n) \sum_{\sigma \in \Sigma_n} \left(n^{1/2} e^{X_\sigma} Y_1^{(\sigma)} \right)^\beta \xrightarrow{d} Y_\beta.$$

The proof of this result is more technical (see Section 7 below) and applies the generating function techniques used in [44]. With more work one could remove the bounded sequence $c(n)$, but this formulation is sufficient for the analysis of the local behavior of μ_1 carried out in Section 8.

Next we turn to the KPZ formula relating the Hausdorff dimension of fractals in $[0, 1]$ in the Euclidean metric and their dimension under a random metric. Specifically, for $\beta \in (0, 1]$ define on $[0, 1]$ the random metric $\rho_\beta(x, y) = \mu_\beta([y, x])$ for $0 \leq x \leq y \leq 1$.

For KPZ relations we replace (2) by the weaker condition

$$(19) \quad \mathbb{E} \xi^2 e^\xi < \infty.$$

The following result is an extension of the result by Benjamini and Schramm [13] on $\beta < 1$ to the critical case $\beta = 1$.

Theorem 6. *Suppose that $\phi(-s) > -\infty$ for all $s \in (0, 1/2)$. Let $K \subset [0, 1]$ be some (deterministic) nonempty Borel set, let ζ_0 denote its Hausdorff dimension with respect to the Euclidean metric, and let ζ denote its Hausdorff dimension with respect to the random metric ρ_1 . Then a.s. ζ is the unique solution of the equation*

$$(20) \quad \zeta_0 = \phi(\zeta)$$

in $[0, 1]$.

In Section 5 we also extend the KPZ relation to the supercritical case $\beta > 1$. In that case, since the measure μ_β is discrete, ρ_β is not a metric anymore; nevertheless μ_β can be formally used in the same way as if ρ_β were a metric to define the Hausdorff dimension of sets K which do not contain any atom of μ_β almost surely. In that case the Hausdorff dimension of K relative to μ_β is the unique solution of $\zeta_0 = \phi(\beta\zeta)$ almost surely, Theorem 12.

We refer to Section 6 for precise statements on the multifractal spectra of the measures μ_β in case $\beta \geq 1$. Finally, in Section 8 we consider the almost sure μ_1 -almost everywhere local behavior of μ_1 in the case of a Gaussian ξ .

We close this section by comparing the phase transition of the cascade measures to that of the Gibbs measures of Random Energy Models (REM's). In Derrida's REM [22] the X_σ , $\sigma \in \Sigma_N$ are taken i.i.d. Gaussian with variance N (in our normalization where $\beta_c = 1$) and $\mu_{\beta,N}$ is normalized to be a probability measure i.e. one considers the Gibbs measure $\tilde{\mu}_{\beta,N}(I_\sigma) = \tilde{Z}_{\beta,N}^{-1} e^{\beta X_\sigma}$ with $\tilde{Z}_{\beta,N} = \sum_{\sigma} e^{\beta X_\sigma}$. REM is a simple model of a disordered

spin system where X_σ is the (random) energy of the spin configuration $\sigma \in \{0, 1\}^N$. In REM the energies are independent whereas in the cascade model they are strongly correlated.

REM has a freezing transition very similar to the cascade model. In [17] it is proven that for $\beta \leq \beta_c$, $\tilde{\mu}_{\beta,N} \rightarrow \tilde{\mu}_\beta$ almost surely as $N \rightarrow \infty$ and the limit measure $\tilde{\mu}_\beta$ is the Lebesgue measure. For $\beta > \beta_c$ [17] prove the analogue of (13) namely that $\tilde{\mu}_{\beta,N}$ converges in distribution to $\tilde{\mu}_\beta$ given by $\tilde{\mu}_\beta([0, t]) \stackrel{d}{=} L_{\frac{1}{\beta}}(t)/L_{\frac{1}{\beta}}(1)$. This result is the REM analogue of (13) since $\tilde{\mu}_1[0, t] = t$. It is actually known (see [18], [1]) that the Gibbs weights in REM and cascade when ordered in decreasing size converge to the same Poisson-Dirichlet process. However, the result (13) is more general as it also gives locations of the atoms.

3. PROOF OF THEOREM 1

Before the proof of Theorem 1 we restate the result on the deterministic normalization that is needed to make the critical case partition function $Z_{1,n}$ converge to a nontrivial random variable in the $n \rightarrow \infty$ limit.

Theorem 7 (Aïdékon and Shi [2]). *Assume that $\mathbb{E} \xi^2 e^\xi < \infty$. Then*

$$n^{1/2} Z_{1,n} = n^{1/2} \sum_{\sigma \in \Sigma_n} e^{X_\sigma} \xrightarrow{\mathbb{P}} Y_1 \quad \text{as } n \rightarrow \infty,$$

where almost surely $0 < Y_1 < \infty$.

We also need the following result of Buraczewski [20] that generalizes Guivarch's tail asymptotics of subcritical Mandelbrot cascades [28] to the critical case.

Theorem 8 (Buraczewski [20]). *Assume (2) and that the distribution of ξ is nonlattice. Then the distribution function $h(x) := \mathbb{P}(Y_1 > x)$ satisfies*

$$h(x) \sim \frac{d}{x} \quad \text{as } x \rightarrow \infty,$$

i.e. there is a positive constant d such that $\lim_{x \rightarrow \infty} xh(x) = d$. If the distribution of ξ is lattice (i.e. supported on some arithmetic sequence), one has $0 < d_1 \leq xh(x) \leq d_2 < \infty$ for large x .

Proof of Theorem 1. We prove (14) first. Fix $\gamma \in (0, 1/2)$ and let $\delta_1 > 0$. By Theorem 8 there exists $x_1 > 0$ such that $h(x) \leq (d + \delta_1)/x \leq 1/2$ for all $x > x_1$, and Theorem 7 shows that

$$(21) \quad \mathbb{P}(\mathcal{B}_n) \rightarrow 1 \quad \text{as } n \rightarrow \infty, \quad \text{where } \mathcal{B}_n := \left\{ \max_{\sigma \in \Sigma_n} e^{X_\sigma} \leq n^{-\gamma}/x_1 \right\}.$$

Recall that by (12) the law of the measure μ_1 satisfies

$$(\mu_1(I_\sigma))_{\sigma \in \Sigma_n} \stackrel{d}{=} \left(e^{X_\sigma} Y_1^{(\sigma)} \right)_{\sigma \in \Sigma_n},$$

where $\{Y_1^{(\sigma)}\}_{\sigma \in \Sigma_n}$ is a family of independent copies of Y_1 , also independent from $\{X_\sigma\}_{\sigma \in \Sigma_n}$. Using this independence we may compute

$$\begin{aligned} \mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma} \mid \{X_\sigma\}\right) &= \prod_{\sigma \in \Sigma_n} \left(1 - \mathbb{P}\left(Y_1^{(\sigma)} \geq n^{-\gamma} e^{-X_\sigma} \mid \{X_\sigma\}\right)\right) \\ &= \prod_{\sigma \in \Sigma_n} \left(1 - h(n^{-\gamma} e^{-X_\sigma})\right). \end{aligned}$$

By taking expectations this gives

$$(22) \quad \mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) = \mathbb{E} \prod_{\sigma \in \Sigma_n} \left(1 - h(n^{-\gamma} e^{-X_\sigma})\right).$$

We employ the elementary inequality $1 - x \geq e^{-2x}$ for $x \in [0, 1/2]$ and Theorem 8 to obtain

$$(23) \quad \begin{aligned} \mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) &\geq \mathbb{E} \mathbf{1}_{\mathcal{B}_n} \prod_{\sigma \in \Sigma_n} \left(1 - h(n^{-\gamma} e^{-X_\sigma})\right) \\ &\geq \mathbb{E} \mathbf{1}_{\mathcal{B}_n} \exp\left(-2(d + \delta_1)n^\gamma \sum_{\sigma \in \Sigma_n} e^{X_\sigma}\right). \end{aligned}$$

By Theorem 7 we have $n^\gamma \sum_{\sigma \in \Sigma_n} e^{X_\sigma} \rightarrow 0$ in probability. By using this fact together with (21) and the bounded convergence theorem we deduce that

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

This proves (14).

To prove (15) we let $\gamma > 1/2$ and use the estimate $1 - x \leq e^{-x}$ for $x \geq 0$ in (22) to get

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) \leq \mathbb{E} \exp\left(-\sum_{\sigma \in \Sigma_n} h(n^{-\gamma} e^{-X_\sigma})\right).$$

We deduce from (6), or alternatively from [44] in the Gaussian case, that $n^{\frac{3}{2}-\varepsilon} \max_{\sigma \in \Sigma_n} e^{X_\sigma} \xrightarrow[n \rightarrow \infty]{d} 0$ for any $\varepsilon > 0$, whence

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} \leq n^{-\frac{3}{2}+\varepsilon}\right) \xrightarrow[n \rightarrow \infty]{} 1.$$

Especially it follows that if $\gamma \in (1/2, 1)$ we have

$$\mathbb{P}(\mathcal{A}_n) \xrightarrow[n \rightarrow \infty]{} 1 \quad \text{for } \mathcal{A}_n := \left\{n^{1/2} < n^{-\gamma} \min_{\sigma \in \Sigma_n} e^{-X_\sigma}\right\}.$$

By Theorem 8, we may find $\delta_2 \in (0, d)$ and $x_2 > 0$ such that $h(x) > (d - \delta_2)/x$ for all $x > x_2$. Consequently, we have

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) \leq \mathbb{E} \mathbf{1}_{\mathcal{A}_n} \exp\left(-(d - \delta_2)n^\gamma \sum_{\sigma \in \Sigma_n} e^{X_\sigma}\right) + (1 - \mathbb{P}(\mathcal{A}_n))$$

for all large n . Since $\gamma > 1/2$, Theorem 7, the bounded convergence theorem, and the fact that $\mathbb{P}(\mathcal{A}_n) \rightarrow 1$ imply that

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{X_\sigma} Y_1^{(\sigma)} < n^{-\gamma}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This concludes the proof of (15) for $\gamma \in (1/2, 1)$, and after this the case $\gamma \geq 1$ is obvious. \square

4. PROOFS OF THEOREMS 3 AND 4.

In this section we employ the notations used in the proof of Theorem 1. In addition, we denote

$$(24) \quad S_{n,1} := n^{1/2} \sum_{\sigma \in \Sigma_n} e^{X_\sigma} \quad \text{and} \quad S_{n,\theta} := n^{\frac{3\theta}{2}} \sum_{\sigma \in \Sigma_n} e^{\theta X_\sigma} \quad \text{for } \theta > 1.$$

For all $\theta \geq 1$ and $\varepsilon > 0$ we shall need the estimates

$$(25) \quad \mathbb{P}(S_{n,\theta} > \lambda) \leq C_{\theta,\varepsilon} \lambda^{-(1-\varepsilon)/\theta} \quad \text{with } C_{\theta,\varepsilon} \text{ independent of } n \geq 1.$$

In the case $\theta > 1$ this estimate is stated explicitly as Proposition 2.1 in [36] for general ξ satisfying $\mathbb{E}\xi^3 e^\xi < \infty$. For $\theta = 1$ we make the stronger assumptions (2) and $\mathbb{E}e^{-h\xi} < \infty$ for some $h > 0$. In this case, Theorem 1.5 of Hu and Shi ([31]) states that for any $\varepsilon \in (0, 1)$ there exists a constant $C_{1,\varepsilon} > 0$ such that

$$(26) \quad \mathbb{E}(S_{n,1})^{1-\varepsilon} \leq C_{1,\varepsilon} \quad \text{for all } n \geq 1.$$

By Chebyshev's inequality the tail estimate (25) follows immediately.

In the case of a Gaussian ξ we give, in Section 7, another approach to establishing the estimate (25), perhaps simpler than the one used in [36] and [31]. By using the generating function techniques of [44] we will study the modulus of continuity of the Laplace transform of $S_{n,\theta}$ at 0 and prove the following lemma.

Lemma 9. *Suppose ξ is Gaussian. Let $\theta \geq 1$ and denote the Laplace transform of $S_{n,\theta}$ by*

$$(27) \quad \phi_{n,\theta}(t) := \mathbb{E} \exp(-t S_{n,\theta}).$$

Then for all $\varepsilon > 0$ there is a constant $C_{\theta,\varepsilon} > 0$, independent of n , such that

$$(28) \quad 1 - \phi_{n,\theta}(t) \leq C_{\theta,\varepsilon} t^{(1-\varepsilon)/\theta} \quad \text{for all } t > 0.$$

The uniform estimate (25) then follows from Chebyshev's inequality and the formula

$$\mathbb{E}(S_{n,\theta})^{(1-\varepsilon)/\theta} = c_{\theta,\varepsilon} \int_0^\infty t^{-1-\frac{1-\varepsilon}{\theta}} (1 - \phi_{n,\theta}(t)) \, dt,$$

where the explicit expression for the constant is $c_{\theta,\varepsilon} = \frac{\theta}{1-\varepsilon} \Gamma(1 - \frac{1-\varepsilon}{\theta})$.

The following lemma contains the essential probability estimate needed for the theorems.

Lemma 10. *Let $\beta \in (0, 1]$. In the case $\beta = 1$ assume that ξ satisfies (2) and $\mathbb{E}e^{-h\xi} < \infty$ for some $h > 0$, and in the case $\beta \in (0, 1)$ assume the same properties as in Theorem 4. Denote $\alpha_1 = 1/2$ and $\alpha_\beta = 3/2$ for $\beta < 1$, and let $\gamma \in (0, \alpha_\beta)$. For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ depending only on β and ε such that*

$$\mathbb{P}\left(\max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) \geq 2^{-n\tilde{\phi}(\beta)} n^{-\gamma\beta}\right) \leq C_\varepsilon n^{(1-\varepsilon)(\gamma-\alpha_\beta)}.$$

Epecially, $2^{n\tilde{\phi}(\beta)} n^{\gamma\beta} \max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) \rightarrow 0$ in probability as $n \rightarrow \infty$.

Proof. Fix $\beta \in (0, 1]$, $\alpha_\beta \in \{1/2, 3/2\}$, $\gamma \in (0, \alpha_\beta)$ and $\varepsilon > 0$ as in the statement of the lemma. Recall from (8) that the distribution of the measure μ_β is given by

$$(\mu_\beta(I_\sigma))_{\sigma \in \Sigma_n} \stackrel{d}{=} \left(2^{-n\tilde{\phi}(\beta)} e^{\beta X_\sigma} Y_\beta^{(\sigma)}\right)_{\sigma \in \Sigma_n},$$

where $\{Y_\beta^{(\sigma)}\}$ is an independent collection of copies of the total mass variable Y_β . Denote $h_\beta(x) = \mathbb{P}(Y_\beta \geq x)$. In the case $\beta = 1$ it follows from the theorem of Buraczewski (Theorem 8) and in the case $\beta < 1$ from our assumptions and the work of Guivarch [28] that there exist constants $x_\beta, d_\beta > 0$ such that

$$(29) \quad h_\beta(x) \leq \frac{d_\beta}{x^{q_\beta}} \leq \frac{1}{2} \quad \text{for all } x > x_\beta,$$

where $q_1 = 1$ and q_β is defined as in Theorem 4 (q_β is necessarily unique due to the strict convexity of $\tilde{\phi}$ and the fact that $q = 1$ is another solution of $\tilde{\phi}(\beta q) - q\tilde{\phi}(\beta) = 0$). Define the events

$$\mathcal{B}_{n,\beta} = \left\{ \min_{\sigma \in \Sigma_n} n^{-\gamma\beta} e^{-\beta X_\sigma} > x_\beta \right\} = \left\{ \max_{\sigma \in \Sigma_n} e^{\beta X_\sigma} < n^{-\gamma\beta} / x_\beta \right\}.$$

Since

$$\mathcal{B}_{n,\beta} \supset \left\{ \sum_{\sigma \in \Sigma_n} e^{\beta q_\beta X_\sigma} < n^{-\gamma\beta q_\beta} / x_\beta^{q_\beta} \right\},$$

by the estimate (25) we have (noticing that $\beta q_\beta > 1$ since $0 \leq \tilde{\phi}(1) \leq \tilde{\phi}(q') < \tilde{\phi}(\beta)$ for all $q' \in (\beta, 1]$)

$$(30) \quad 1 - \mathbb{P}(\mathcal{B}_{n,\beta}) \leq \mathbb{P}\left(S_{n,\beta q_\beta} \geq n^{(\alpha_\beta - \gamma)\beta q_\beta} / x_\beta^{q_\beta}\right) \leq C n^{(1-\varepsilon)(\gamma - \alpha_\beta)}$$

for some constant $C > 0$ depending on β and ε .

Using the estimates (29) we may perform a computation similar to the one used to obtain (23) in the proof of Theorem 1. The resulting estimate is

$$\begin{aligned} \mathbb{P}\left(\max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) < 2^{-n\tilde{\phi}(\beta)} n^{-\gamma\beta}\right) &= \mathbb{P}\left(\max_{\sigma \in \Sigma_n} e^{\beta X_\sigma} Y_\beta^{(\sigma)} < n^{-\gamma\beta}\right) \\ &= \mathbb{E} \prod_{\sigma \in \Sigma_n} \left(1 - h_\beta\left(n^{-\gamma\beta} e^{-\beta X_\sigma}\right)\right) \\ &\geq \mathbb{E} \mathbf{1}_{\mathcal{B}_{n,\beta}} \exp\left(-2d_\beta \sum_{\sigma \in \Sigma_n} n^{\gamma\beta q_\beta} e^{\beta q_\beta X_\sigma}\right), \end{aligned}$$

which in combination with (30) and (25) yields

$$\begin{aligned}
\mathbb{P} \left(\max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) \geq 2^{-n\tilde{\phi}(\beta)} n^{-\gamma\beta} \right) \\
\leq 1 - \mathbb{P}(\mathcal{B}_{n,\beta}) + 1 - \mathbb{E} \exp \left(-2d_\beta \sum_{\sigma \in \Sigma_n} n^{\gamma\beta q_\beta} e^{\beta q_\beta X_\sigma} \right) \\
= 1 - \mathbb{P}(\mathcal{B}_{n,\beta}) + 1 - \mathbb{E} \exp \left(-2d_\beta n^{(\gamma-\alpha_\beta)\beta q_\beta} S_{n,\beta q_\beta} \right) \\
\leq C' n^{(1-\varepsilon)(\gamma-\alpha_\beta)},
\end{aligned}$$

where the constant $C' > 0$ depends only on β and ε . The proof of the lemma is complete. \square

Theorems 3 and 4 now follow easily.

Proof of Theorem 4. Let $\beta \in (0, 1)$ and $\gamma \in (0, 1/2)$. As it is clearly enough to consider dyadic intervals in (17), all we need to do is to improve the convergence in probability in Lemma 10 to almost sure convergence. But since $\gamma - 3/2 < -1$, we may take an $\varepsilon > 0$ small enough that

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) \geq 2^{-n\tilde{\phi}(\beta)} n^{-\gamma\beta} \right) \leq C_\varepsilon \sum_{n=1}^{\infty} n^{(1-\varepsilon)(\gamma-\frac{3}{2})} < \infty.$$

The Borel–Cantelli lemma now implies the existence of a random constant $C(\omega) < \infty$ such that

$$\max_{\sigma \in \Sigma_n} \mu_\beta(I_\sigma) < C(\omega) 2^{-n\tilde{\phi}(\beta)} n^{-\gamma\beta} \quad \text{for all } n = 1, 2, \dots,$$

which is the desired conclusion. \square

Proof of Theorem 3. Let $\gamma \in (0, 1/2)$ and choose an integer $\ell \geq 1$ so that $\ell(\gamma - 1/2) < -2$. Applying Lemma 10 with $\varepsilon = 1/2$ gives

$$\sum_{k=1}^{\infty} \mathbb{P} \left(\max_{\sigma \in \Sigma_{k^\ell}} \mu_1(I_\sigma) \geq (k^\ell)^{-\gamma} \right) \leq C_{1/2} \sum_{k=1}^{\infty} k^{\ell \frac{\gamma-1/2}{2}} < \infty.$$

Borel–Cantelli lemma then implies

$$\max_{\sigma \in \Sigma_{k^\ell}} \mu_1(I_\sigma) \leq C'(\omega) (k^\ell)^{-\gamma} \quad \text{for a random constant } C'(\omega) < \infty$$

for the dyadic intervals of levels $1^\ell, 2^\ell, 3^\ell, \dots$. It remains to note that the sequence of maxima $(\max_{\sigma \in \Sigma_n} \mu_1(I_\sigma))_{n=1}^\infty$ is decreasing, so for $k^\ell \leq n < (k+1)^\ell$ we have

$$\max_{\sigma \in \Sigma_n} \mu_1(I_\sigma) \leq \max_{\sigma \in \Sigma_{k^\ell}} \mu_1(I_\sigma) \leq C'(\omega) k^{-\ell\gamma} < C'(\omega) 2^{\ell\gamma} n^{-\gamma}.$$

This shows that the estimate (16) indeed holds. Finally, the statement that one cannot take $\gamma > 1/2$ in the result is an immediate consequence of the divergence result (15) in Theorem 1. \square

5. KPZ RELATIONS

5.1. The KPZ formula associated with μ_1 . In the case where ξ is Gaussian the relation stated by the Theorem 6 is precisely the KPZ formula predicted by physicists working in quantum gravity. Benjamini and Schramm [13] proved Theorem 6 for the random metrics defined by the one-dimensional Mandelbrot cascade measures, i.e. in the subcritical case

$$(31) \quad \mathbb{E}e^\xi = \frac{1}{2} \quad \text{and} \quad \mathbb{E}\xi e^\xi < 0,$$

which basically corresponds to the measures μ_β for $\beta < 1$ considered in Section 2. They were inspired by a different point of view developed by Duplantier and Sheffield [24], who gave a sense to the KPZ formula in terms of expected box counting dimension in the context of Liouville quantum gravity, by considering random measures associated with the Gaussian free field.

In addition, Rhodes and Vargas [42] derived in dimension 1 a relation similar to (20) between Hausdorff dimensions when comparing Euclidean geometry and the geometry given by the random metric associated with the limit measure of a non-degenerate infinitely divisible cascade. In higher dimension they obtained such a formula by using the Lebesgue measure and the random measure, not as metric, but as functions of balls to define Hausdorff measures and dimension (notice that in the log-Gaussian case, the multiplicative chaos of [42] and the measures considered in [24] are closely related).

Before starting the proof of Theorem 6, let us first comment on the difference between our assumptions on moments of negative orders of e^ξ and those made in [13] in the subcritical case. If we denote by $\tilde{\mu}_1$ the measure $\tilde{\mu}_1 = 2e^{\tilde{\xi}}\mu_1$, where $\tilde{\xi}$ is a copy of ξ independent of μ_1 , then $\tilde{Y}_1 = \|\tilde{\mu}_1\|_{\text{TV}}$ satisfies the same functional equation as the total mass of the Mandelbrot measure considered in [13]. Then it follows from [13] that for $s > 0$ we have $\mathbb{E}\tilde{Y}_1^{-s} < \infty$ if and only if $\phi(-s) > -\infty$, and for the proof of KPZ formulas one needs $\mathbb{E}\tilde{Y}_1^{-s} < \infty$ for all $s \in (0, 1)$. Due to our definition of μ_1 , setting $Y_1 = \|\mu_1\|_{\text{TV}}$ we have [40, Theorem 4(a)]

$$(32) \quad \mathbb{E}Y_1^{-s} < \infty \quad (\forall s \in (0, 1)) \quad \text{as soon as} \quad \phi(-s/2) > -\infty \quad (\forall s \in (0, 1/2))$$

where $\phi(s) = -\log_2 \mathbb{E}(e^{s\xi})$. Moreover, $\mathbb{E}(Y_1^s) < \infty$ for $s \in (0, 1)$, see [15, Theorem 4 and 5].

Proof of Theorem 6. After the above observations, the proof of the KPZ formula established in [13] can be mimicked, up to changes imposed by the fact that $\mathbb{E}(Y_1^s) < \infty$ for $s \in (0, 1)$ but not for $s = 1$, and that we make precise below.

Employing the notation of [13], let us denote by ℓ the total mass Y_1 of μ_1 . We use the following form of Lemma 3.3 in [13]: Let $x, y \in [0, 1]$ and let $s \in (0, 1)$. Then $\mathbb{E}(\rho_1(x, y)^s) \leq 8|x - y|^{\phi(s)}\mathbb{E}(\ell^s)$.

For the upper bound for the Hausdorff dimension, which corresponds to [13, Theorem 3.4], due to the above analogue of [13, Lemma 3.3] the proof is the same in case $\zeta_0 < 1$. In turn, the case $\zeta_0 = 1$ is trivial.

For the lower bound, using the same notations as in [13], we fix a non-empty Borel set K such that $\zeta_0 > 0$ and $t \in (0, \zeta_0)$. We set $s = \phi^{-1}(t)$, denote by ν_0 a positive Borel measure carried by K and such that $\mathcal{E}_t(\nu_0) = \int \int \frac{\nu_0(dx)\nu_0(dy)}{|y-x|^t} < \infty$, and consider the sequence of measures $(\nu_n)_{n \geq 1}$ whose densities with respect to ν_0 are given by $e^{sX_\sigma}/(\mathbb{E}e^{s\xi})^{|\sigma|}$ over each interval I_σ . Here $|\sigma| = n$ for $\sigma \in \Sigma_n$.

Also, we consider $\rho_{1,n}(x, y) = \max(\rho_1(x, y), \mu_1(I_n(x)), \mu_1(I_n(y)))$ for all $x, y \in [0, 1]$, where $I_n(x)$ stands for the closure of the semi-open to the left dyadic interval of generation n containing x , and $[1 - 2^{-n}, 1]$ if $x = 1$. Then, following [13, Theorem 3.5] proof, we get

$$\mathbb{E}(\mathcal{E}_s(\nu_n, \rho_{1,n}) = \int \int \frac{\nu_n(dx)\nu_n(dy)}{\rho_{1,n}(x, y)^s}) = O(1)\mathcal{E}_t(\nu_0).$$

Then, noting that $\nu_n([0, 1])^2 \ell^{-s} \leq \mathcal{E}_s(\nu_n, \rho_{1,n})$, at the end of the proof of [13, Theorem 3.5], Hölder's inequality must be applied as follows, with $h \in (1 + s, 2)$:

$$\begin{aligned} \mathbb{E}[\nu_n([0, 1])^{h/(1+s)}] &= \mathbb{E}\left[\left(\nu_n([0, 1])^{h/1+s} \ell^{-hs/2(1+s)}\right) \ell^{hs/2(1+s)}\right] \\ &\leq \mathbb{E}\left[\nu_n([0, 1])^h \ell^{-hs/2}\right]^{1/(1+s)} \mathbb{E}(\ell^{h/2})^{s/(1+s)} \leq \mathbb{E}\left[\nu_n([0, 1])^2 \ell^{-s}\right]^{h/2(1+s)} \mathbb{E}(\ell^{h/2})^{s/(1+s)} \\ &= O(1)(\mathcal{E}_t(\nu_0))^{h/2(1+s)} \mathbb{E}(\ell^{h/2})^{s/(1+s)}, \end{aligned}$$

which is finite because $\mathcal{E}_t(\nu_0) < \infty$ and $\mathbb{E}(\ell^{h/2})^{s/(1+s)} < \infty$ since $h/2 < 1$. Finally, the martingale $\nu_n([0, 1])$ is bounded in $L^{h/(1+s)}$, so it is uniformly integrable since $h/(1+s) > 1$, and ν_n converges weakly almost surely to a nondegenerate measure ν , necessarily supported on K , and such that $\int \int \frac{\nu(dx)\nu(dy)}{\rho_1(x, y)^s} < \infty$, which implies that the lower Hausdorff dimension of ν with respect to ρ_1 is at least s . Thus, the Hausdorff dimension of K is almost surely at least s for all $s < \zeta$, hence the conclusion. \square

5.2. The KPZ formula associated with μ_β . Assume that $\alpha \in (0, 1)$ and let L_α be a stable subordinator of index α independent of the σ -algebra generated by $\{\xi_\sigma : \sigma \in \Sigma\}$. We recall that up to a multiplicative constant μ_β is the measure obtained as the derivative of the function $L_{1/\beta} \circ F_{\mu_1}$ on $[0, 1]$ where $F_{\mu_1}(x) = \mu_1([0, x])$. It is also of interest to consider the measures obtained in the same way from a subcritical cascade measure μ . Altogether these measures unify stable Lévy subordinators and Mandelbrot measures in a natural class of generalized semi-stable processes which satisfy scaling properties similar to (12).

Let us fix the conventions used in this subsection and in Section 6 below:

- μ stands for the Mandelbrot cascade measure generated by the variable ξ in the subcritical case (31), so that $\mathbb{E}\xi e^\xi < 0$. Then we denote by ν_α the measure obtained as the derivative of $L_\alpha \circ F_\mu$.
- As before μ_1 stands for a critical Mandelbrot measure, whence $\mathbb{E}\xi e^\xi = 0$. We denote by $\nu_{\alpha,1}$ the measure obtained as the derivative of $L_\alpha \circ F_{\mu_1}$. Thus, up to a positive multiplicative constant, we have $\mu_\beta = \nu_{1/\beta,1}$ for $\beta > 1$.

A natural way to extend the usual notion of box-counting and Hausdorff dimension is to replace the metric by a continuous measure [16, p. 141]. This is what was used in [24] and [42] respectively, to get the KPZ relations in dimension ≥ 2 . Thus, if ν is a positive continuous Borel measure supported on $[0, 1]$, we can define for $s \geq 0$ and any subset E of $[0, 1]$

$$H_\nu^s(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i \geq 1} \nu(I_i)^s : E \subset \bigcup_{i \geq 1} I_i, I_i \text{ interval of length } \leq \delta \right\},$$

and the Hausdorff dimension of E relative to ν as

$$(33) \quad \dim_\nu(E) = \sup\{s \geq 0 : H_\nu^s(E) = \infty\} = \inf\{s \geq 0 : H_\nu^s(E) = 0\}.$$

Note that since we are in dimension 1, this definition equals the definition of the Hausdorff dimension in the metric

$$\rho_\nu(x, y) := \nu([x, y]).$$

If the measure ν is not continuous, it is easy to check that (33) is still defined if $\nu(E) = 0$. Thus we can seek for analogues of the KPZ formula invoking Hausdorff dimensions relative to ν_α or $\nu_{\alpha,1}$. This will use the following lemma.

Lemma 11. (1) *If E is a Borel set of null Lebesgue measure, then $\mu(E) = 0$ and $\nu_\alpha(E) = 0$ a.s.*
 (2) *Suppose that $\mathbb{E} \xi^2 e^\xi < \infty$. If E is a Borel set of Euclidean Hausdorff dimension less than 1, then $\mu_1(E) = 0$ and $\nu_{\alpha,1}(E) = 0$ a.s.*

Proof. We only prove (2) since (1) is similar and slightly simpler. Let $1 > t > \dim K$, $\epsilon > 0$, and consider a covering of E by a collection $(I_i)_{i \geq 1}$ of dyadic subintervals of $[0, 1]$ such that $\sum_{i \geq 1} |I_i|^t \leq \epsilon$.

The random variable $\sum_{i \geq 1} \nu_{\alpha,1}(I_i)$ is equal in distribution to $\sum_{i \geq 1} \mu_1(I_i)^{1/\alpha} Z_i$, where the random variables Z_i are identically distributed with a positive α -stable random variable Z , and independent of μ_1 . It follows that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i \geq 1} \nu_{\alpha,1}(I_i) \right)^{t\alpha} \right] &\leq \sum_{i \geq 1} \mathbb{E}(\mu_1(I_i)^t) \mathbb{E}(Z^{t\alpha}) \\ &= \sum_{i \geq 1} (\mathbb{E} e^{t\xi})^{-\log_2 |I_i|} \mathbb{E}(Y_1^t) \mathbb{E}(Z^{t\alpha}) \\ &\leq \sum_{i \geq 1} (\mathbb{E} e^\xi)^{-t \log_2 |I_i|} \mathbb{E}(Y_1^t) \mathbb{E}(Z^{t\alpha}) \\ &= \mathbb{E}(Y_1^t) \mathbb{E}(Z^{t\alpha}) \sum_{i \geq 1} |I_i|^t \leq \mathbb{E}(Y_1^t) \mathbb{E}(Z^{t\alpha}) \epsilon. \end{aligned}$$

Taking $\epsilon = 2^{-n}$ we get a deterministic sequence of coverings $(\bigcup_{i \geq 1} I_i^n)_{n \geq 1}$ of E such that a.s. $\sum_{n \geq 1} \left(\sum_{i \geq 1} \nu_{\alpha,1}(I_i^n) \right)^{t\alpha} < \infty$, hence $\lim_{n \rightarrow \infty} \sum_{i \geq 1} \nu_{\alpha,1}(I_i^n) = 0$.

A similar calculation shows that $\mu_1(E) = 0$ a.s. □

We can now state a result regarding the KPZ relations associated with ν_α or $\nu_{\alpha,1}$.

Theorem 12. *Let $\alpha \in (0, 1)$. Suppose that the variable ξ satisfies $\phi(-s) > -\infty$ for all $s \in (0, 1/2)$, where ϕ is defined as before by (9). Let $K \subset [0, 1]$ be some (deterministic) nonempty Borel set and let ζ_0 denote its Hausdorff dimension with respect to the Euclidean metric.*

- (1) *Suppose that K has Lebesgue measure 0. Let ζ denote the Hausdorff dimension of K with respect to the random metric ρ_μ and ζ_α its Hausdorff dimension relative to ν_α . Then a.s. ζ_α is the unique solution of the equation*

$$\zeta_0 = \phi(\zeta_\alpha/\alpha)$$

in $[0, \alpha]$, i.e. $\zeta_\alpha = \alpha\zeta$.

- (2) *Suppose that $\mathbb{E} \xi^2 e^\xi < \infty$ and $\zeta_0 < 1$. Let ζ denote the Hausdorff dimension of K with respect to the random metric ρ_{μ_1} and ζ_α its Hausdorff dimension relative to $\nu_{\alpha,1}$. The same conclusion as in part (1) holds.*

An analogue to Theorem 12.1 was first proved in [7] in the context of non-degenerate Kahane Gaussian multiplicative chaos, and it gave a rigorous mathematical justification to the so called dual KPZ formula.

Proof. Note that computing ζ , the Hausdorff dimension of K relative to μ (resp. μ_1) amounts to computing the Euclidean Hausdorff dimension of the image K_μ (resp. K_{μ_1}) of K by F_μ (resp. F_{μ_1}). Moreover, computing ζ_α , the Hausdorff dimension of K relative to ν_α (resp. $\nu_{\alpha,1}$) amounts to computing the Euclidean Hausdorff dimension of $L_\alpha(K_\mu)$ (resp. $L_\alpha(K_{\mu_1})$). Now we can use the fact that a.s. $L_\alpha(E) = \alpha \dim E$ for all subsets E of $[0, 1]$ [14, III.5] to conclude that $\zeta_\alpha = \alpha\zeta$. \square

One may observe that in some sense the effect of combining the Lévy process L_α with the measure μ_1 can be thought as a 'random snowflaking' (with exponent α) of the random metric induced by ν_α , see [29].

6. MULTIFRACTAL NATURE OF THE MEASURES μ_β , $\beta \geq 1$

The knowledge of the continuity of the measure μ_1 has some consequences in the multifractal analysis of Mandelbrot measures and the multifractal analysis of Lévy processes in multifractal time.

Recall that given a positive Borel measure ν supported on a compact metric space (X, d) , its multifractal analysis consists in computing the Hausdorff dimension of the level sets of the pointwise Hölder exponent of ν , namely the sets

$$E_\nu(\gamma) = \left\{ x \in X : \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(x, r))}{\log(r)} = \gamma \right\} \quad (\gamma \in [0, \infty]).$$

Throughout, we adopt the convention that a set has a negative dimension if and only if it is empty.

Now let μ (resp. μ_1) be the subcritical (resp. critical) measure considered in section 5.2 and set $\tau(s) := \phi(s) - 1 = -\tilde{\phi}(s)$. Under suitable assumptions, the multifractal nature of the Mandelbrot measure μ has been studied in [30, 40], in which it is shown that for each γ such that $\tau^*(\gamma) > 0$ one has almost surely $\dim E_\mu(\gamma) = \tau^*(\gamma)$ (in these papers the pointwise Hölder

exponent is associated with the dyadic intervals rather than centered balls). Here τ^* stands for the Legendre transform

$$\tau^*(\gamma) := \inf_{t \in \mathbb{R}} (t\gamma - \tau(t)).$$

This result is strengthened in [4]: almost surely (simultaneously) for all γ such that $\tau^*(\gamma) > 0$ it holds that $\dim E_\mu(\gamma) = \tau^*(\gamma)$.

Moreover, in [4] the question for the at most two values of γ for which $\tau^*(\gamma) = 0$ is partially solved. Here one employs the fact that in this case γ takes the form $\tau'(s_0)$, and one verifies that $E_\mu(\tau'(s_0))$ carries a piece of a critical measure of $\tilde{\mu}_1$, where $\tilde{\mu}_1$ is constructed by using instead the normalized variable $e^{\tilde{\xi}} := \frac{e^{s_0\xi}}{2\mathbb{E}e^{s_0\xi}}$. In particular, this set is nonempty, but its Hausdorff dimension equals 0. The fact that we now know that $\tilde{\mu}_1$ is atomless makes it possible to strengthen the above result: $E_\mu(\tau'(s_0))$ is not countable.

The results of [4] hold also for the critical measures μ_1 . Then in [6] in which limit of complex multiplicative cascades are studied, a complete answer was given for the multifractal behaviour of μ , and because we now know that μ_1 is atomless they extend easily to μ_1 , and we may state the following result without proof.

Theorem 13. *Let $\nu = \mu$ or $\nu = \mu_1$ according to $\mathbb{E}e^\xi = \frac{1}{2}$ and $\mathbb{E}\xi e^\xi < 0$ or $\mathbb{E}e^\xi = \frac{1}{2}$ and $\mathbb{E}\xi e^\xi = 0$. Suppose that $\phi(s) > -\infty$ for all $s \in \mathbb{R}$ if $\nu = \mu$, and $\phi(s) > -\infty$ in a neighborhood of $(-\infty, 1]$ if $\nu = \mu_1$. With probability 1, for all $\gamma \in [0, \infty]$, $E_\nu(\gamma) \neq \emptyset$ if and only if γ belongs to the compact interval $I = \{\gamma : \tau^*(\gamma) = \inf_{s \in \mathbb{R}} (s\gamma - \tau(s)) \geq 0\}$, and in this case $\dim E_\nu(\gamma) = \tau^*(\gamma)$. Moreover, $\min(I) = 0$ if and only if $\nu = \mu_1$.*

The previous results can be extended if $[0, 1]$ is endowed with a random metric associated with a non-degenerate Mandelbrot cascade (as it was done in [3]) or a critical Mandelbrot cascade built simultaneously with μ or μ_1 .

Given $\alpha \in (0, 1)$, the multifractal nature of the measure ν_α associated with L_α and μ as in Section 5.2 has been studied in [12]. The case of $\nu_{\alpha,1}$ was not treated in [12], mainly because of the lack of information on the discrete or continuous nature of μ_1 . After our Theorem 1 it is not hard to adapt the approach developed in [12] to achieve the multifractal analysis of $\nu_{\alpha,1}$. It is even easier than that of ν_α because the difficult discussion associated with the degree of approximations of the points of $[0, 1]$ by the atoms of ν_α is not necessary. Consequently we just state the result:

Theorem 14. *Let $\alpha \in (0, 1)$. Suppose that $\phi(s) > -\infty$ on a neighborhood of $(-\infty, 1)$. Let $\tau_\alpha(s) = \min(\tau(s/\alpha), 0)$. Let $\nu = \nu_\alpha$ or $\nu = \nu_{\alpha,1}$ according to $\mathbb{E}e^\xi = \frac{1}{2}$ and $\mathbb{E}\xi e^\xi < 0$ or $\mathbb{E}e^\xi = \frac{1}{2}$ and $\mathbb{E}\xi e^\xi = 0$. With probability 1, for all $\gamma \in [0, \infty]$, $E_\nu(\gamma) \neq \emptyset$ if and only if γ belongs to the compact interval $I = [0, \gamma_{\max} = \max\{\gamma : \tau_\alpha^*(\gamma) \geq 0\}]$, and in this case $\dim E_\nu(\gamma) = \tau_\alpha^*(\gamma)$.*

In particular, when $\nu = \nu_\alpha$, τ^* is linear of slope α over $[0, \tau'(1)/\alpha]$ and strictly concave over $[\tau'(1)/\alpha, \gamma_{\max}]$, the linear part being reminiscent of the atoms of L_α , while when $\nu = \nu_{\alpha,1}$, such a linear part disappears.

7. PROOFS OF THEOREM 5 AND LEMMA 9

We will prove Theorem 5 using the results in [44]. Let $\beta > 1$ and denote by Z_n the random variable

$$(34) \quad Z_n = \sum_{\sigma \in \Sigma_n} \left(e^{X_\sigma} Y_1^{(\sigma)} \right)^\beta.$$

It will be convenient to define the following reparametrization of its Laplace transform:

$$(35) \quad H_{n,\beta}(x) := \mathbb{E} e^{-e^{\beta x} Z_n}.$$

The convergence of $c(n)n^{\beta/2}Z_n$ in distribution (where $\log c(n)$ is a bounded sequence) will follow as we prove that there is a bounded sequence $C(n)$ so that

$$(36) \quad H_{n,\beta}(x + \log \sqrt{n} + C(n))$$

converges for all $x \in \mathbb{R}$ (to a function with limit 1 at $-\infty$) as $n \rightarrow \infty$.

We start by deriving a recursion relation for $H_{n,\beta}(x)$ in n . Given a non-negative random variable Y we define the random variable $T_\beta Y$ by

$$(37) \quad T_\beta Y \stackrel{d}{=} \left((e^{\xi_0} Y^{(0)})^\beta + (e^{\xi_1} Y^{(1)})^\beta \right)^{\frac{1}{\beta}},$$

where $Y^{(0)}$ and $Y^{(1)}$ are independent copies of Y and ξ_0 and ξ_1 are independent copies of ξ which are also independent of $Y^{(0)}$ and $Y^{(1)}$. With this definition we have

$$(38) \quad Z_n \stackrel{d}{=} (T_\beta^n Y_1)^\beta.$$

Passing to Laplace transforms and using independence, we get the desired recursion:

$$\begin{aligned} H_{n+1,\beta}(x) &= \mathbb{E} \left(\exp \left(-e^{\beta x} (T_\beta^{n+1} Y_1)^\beta \right) \right) \\ &= \mathbb{E} \left(\exp \left(-e^{\beta x} \left((e^{\xi_0} (T_\beta^n Y_1)^{(0)})^\beta + (e^{\xi_1} (T_\beta^n Y_1)^{(1)})^\beta \right) \right) \right) \\ &= \mathbb{E} \left(\exp \left(-e^{\beta x} e^{\beta \xi} (T_\beta^n Y_1)^\beta \right) \right)^2 \\ &= \left(\int \rho(y) H_{n,\beta}(x + y) dy \right)^2, \end{aligned}$$

where ρ is the density of the distribution of ξ . $H_{n,\beta}(x)$ is determined from this recursion given the initial data

$$H_{0,\beta}(x) = \mathbb{E}(\exp(-e^{\beta x} Y_1^\beta)).$$

To make the connection to [44] we restrict to the Gaussian case, i.e. $\xi \sim N(-2 \log 2, 2 \log 2)$, and define

$$G_{n,\beta}(x) = \left(H_{n,\beta}(2n \log 2 - \sqrt{2 \log 2} x) \right)^{\frac{1}{2}}.$$

Some calculation gives the following recursion

$$(39) \quad G_{n+1,\beta}(x) = \int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} (G_{n,\beta}(x+y))^2 dy$$

with initial data

$$(40) \quad G_{0,\beta}(x) = \left(\mathbb{E} e^{-e^{-\beta\sqrt{2\log 2}x} Y_1^\beta} \right)^{\frac{1}{2}}.$$

Thus finding a bounded sequence $C(n)$ such that (36) converges amounts to finding a bounded sequence $C'(n)$ so that

$$(41) \quad G_{n,\beta} \left(x + n\sqrt{2\log 2} - \frac{\log n}{2\sqrt{2\log 2}} + C'(n) \right)$$

converges for all $x \in \mathbb{R}$ to an appropriate limit. The recursion (39) was studied in [44]. The main result is

Theorem 15. (a) Let $G_n^{(\alpha)}$ be given by the recursion (39) with initial data

$$(42) \quad G_0^{(\alpha)}(x) = \exp(-e^{-\alpha x}) \quad 0 < \alpha \leq \infty$$

(where $G_0^{(\infty)}$ is the Heaviside function $\theta(x)$). Let $m_n^{(\alpha)} = \left(G_n^{(\alpha)} \right)^{-1} \left(\frac{1}{2} \right)$.

Then, (as $n \rightarrow \infty$) $G_n^{(\alpha)} \left(x + m_n^{(\alpha)} \right)$ converges uniformly on \mathbb{R} to a function $w^{(\alpha)}$ satisfying

$$(43) \quad w^{(\alpha)}(x) = \int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} w^{(\alpha)}(x+y+c(\alpha))^2 dy,$$

where

$$(44) \quad c(\alpha) = \begin{cases} \frac{\alpha}{2} + \frac{\log 2}{\alpha}, & \text{for } \alpha \leq \sqrt{2\log 2} \\ c(\sqrt{2\log 2}), & \text{for } \alpha > \sqrt{2\log 2} \end{cases}.$$

The function $w^{(\alpha)}$ is the unique solution to this equation under the assumptions that $w^{(\alpha)}(0) = \frac{1}{2}$, $w^{(\alpha)}(-\infty) = 0$, $w^{(\alpha)}(\infty) = 1$ and $w^{(\alpha)}$ is increasing. Moreover, the shift sequence $(m_n^{(\alpha)})$ exhibits the following asymptotic behavior:

$$(45) \quad m_n^{(\alpha)} = \begin{cases} c(\alpha)n + \gamma(\alpha) + o(1), & \text{for } \alpha < \sqrt{2\log 2} \\ \sqrt{2\log 2}n - \frac{1}{2\sqrt{2\log 2}} \log n + \mathcal{O}(1), & \text{for } \alpha = \sqrt{2\log 2} \\ \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}} \log n + \mathcal{O}(1), & \text{for } \alpha > \sqrt{2\log 2} \end{cases},$$

where $\gamma(\alpha)$ is a some real number.

(b) Suppose¹ the initial condition $G_0(x)$ is an increasing function with $G_0(-\infty) = 0$ and $1 - G_0(x) \sim e^{-\alpha x}$ as $x \rightarrow \infty$. If $\alpha < \sqrt{2\log 2}$ the results of (a) hold. Moreover, for $\alpha \geq \sqrt{2\log 2}$, if the sequence $G_n^{(\alpha)} \left(x + m_n^{(\alpha)} \right)$ (defined now using the initial data G_0) converges uniformly then (45) holds.

Proof. For (a) see Theorem 3.4, Corollary 3.5 and sections 4 and 5 in [44]. For (b) $\alpha < \sqrt{2\log 2}$, see Theorem 3.4 and Lemma 5.3 in [44] and for $\alpha \geq \sqrt{2\log 2}$ see the argument in Section 6 of [44]. \square

¹In this section the notation $A(x) \sim B(x)$ as $x \rightarrow \infty$ is a shorthand for $\lim_{x \rightarrow \infty} (A(x)/B(x)) = c$ for some $c \in (0, \infty)$.

To apply this Theorem we need the asymptotics of the initial data $G_{0,\beta}$ in (40). In fact, we claim that

$$(46) \quad 1 - G_{0,\beta}(x) \sim e^{-\sqrt{2\log 2}x} \quad \text{as } x \rightarrow \infty.$$

Indeed, let $\phi_\beta(t) = \mathbb{E}(\exp(-tY_1^\beta))$. Using Theorem 8 we see that as $t \rightarrow 0$

$$(47) \quad 1 - \phi_\beta(t) = t \int_0^\infty e^{-tx} \mathbb{P}(Y_1^\beta > x) dx$$

$$(48) \quad = \int_0^\infty e^{-x} \mathbb{P}(Y_1 > t^{-\frac{1}{\beta}} x^{\frac{1}{\beta}}) dx \sim t^{\frac{1}{\beta}}.$$

Hence $1 - H_{0,\beta}(x) \sim (e^{\beta x})^{\frac{1}{\beta}} = e^x$ as $x \rightarrow -\infty$ which translates to (46) for $G_{0,\beta}(x)$

Thus the convergence of (41) follows from Theorem 15 (b) provided we show the sequence $G_{n,\beta}(x + m_{n,\beta})$ converges with the choice $m_{n,\beta} \equiv (G_{n,\beta})^{-1}(\frac{1}{2})$. We prove the convergence of $G_{n,\beta}(x + m_{n,\beta})$ by comparing it to the solutions provided by Theorem 15 (a) with $\alpha < \sqrt{2\log 2}$ and $\alpha = \infty$ using the following maximum principle from [44]:

Proposition 16. *Let G_n^1 and G_n^2 be given by the recursion (39) with initial data G_0^1 and G_0^2 with the property that there exists a point x_0 so that $G_0^2(x) \geq G_0^1(x)$ for $x \geq x_0$ and $G_0^2(x) \leq G_0^1(x)$ for $x \leq x_0$. Then for every $n \geq 1$ there exists a point x_n so that $G_n^2(x) \geq G_n^1(x)$ for $x \geq x_n$ and $G_n^2(x) \leq G_n^1(x)$ for $x \leq x_n$. The claim holds also with strict inequalities.*

We perform the comparison by considering the following family of initial conditions with $\beta_1 > 0$:

$$(49) \quad G_0^{(\beta_1, \beta)}(x) = \left(\mathbb{E} e^{-e^{-\beta_1 \sqrt{2\log 2}x} Y_1^\beta} \right)^{\frac{1}{2}}.$$

Note that $G_0^{(\beta, \beta)} = G_{0,\beta}$, $G_0^{(\infty, \beta)} = \theta(x)$ and from (47) we have

$$(50) \quad 1 - G_0^{(\beta_1, \beta)}(x) \sim e^{-\frac{\beta_1}{\beta} \sqrt{2\log 2}x} \quad \text{as } x \rightarrow \infty.$$

Let $G_n^{(\beta_1, \beta)}$ be given by the recursion (39) with this initial data and $m_{(\beta_1, \beta), n} = \left(G_n^{(\beta_1, \beta)} \right)^{-1}(\frac{1}{2})$ (one can use the recursion to check that $G_n^{(\beta_1, \beta)}$ is strictly increasing and this is well defined).

Lemma 17. *For $x \geq 0$, $G_n^{(\beta_1, \beta)}(x + m_{(\beta_1, \beta), n})$ is increasing in β_1 and for $x \leq 0$ it is decreasing in β_1 .*

Proof. Let $\beta_1 > \beta'_1$ and n_0 be some fixed positive integer. Define $G_0^2(x) = G_0^{(\beta_1, \beta)}(x + m_{(\beta_1, \beta), n_0})$ and $G_0^1(x) = G_0^{(\beta'_1, \beta)}(x + m_{(\beta'_1, \beta), n_0})$. We then note that if

$$(51) \quad x > x_0 := \frac{\beta'_1 m_{(\beta'_1, \beta), n_0} - \beta_1 m_{(\beta_1, \beta), n_0}}{\beta_1 - \beta'_1},$$

then

(52)

$$\exp\left(-e^{-\beta_1\sqrt{2\log 2}(x+m_{(\beta_1,\beta),n_0})}Y_1^\beta\right) > \exp\left(-e^{-\beta'_1\sqrt{2\log 2}(x+m_{(\beta'_1,\beta),n_0})}Y_1^\beta\right)$$

and $G_0^2(x) > G_0^1(x)$. Similarly for $x < x_0$, $G_0^2(x) < G_0^1(x)$. Thus by Proposition 16 there exists a point x_n so that $G_n^2(x) > G_n^1(x)$ for $x > x_n$, and the opposite inequality holds for $x < x_n$. Let us set $n = n_0$ and note that $G_{n_0}^2(0) = G_{n_0}^1(0) = \frac{1}{2}$. Thus $x_{n_0} = 0$. Since n_0 was arbitrary, we have proven our claim. \square

We can now finish the proof of convergence of the sequence $G_{n,\beta}(x + m_{n,\beta})$. First, by Theorem 15(b) and (50), the quantity $G_n^1(x) := G_n^{(\beta_1,\beta)}(x + m_{(\beta_1,\beta),n})$ converges to $w^{(\alpha)}(x)$ uniformly in x with $\alpha = \frac{\beta_1}{\beta}\sqrt{2\log 2}$ provided $\beta_1 < \beta$. Second, by Theorem 15 (a) $G_n^2(x) := G_n^{(\infty,\beta)}(x + m_{(\infty,\beta),n})$ converges to $w^{(\infty)}(x) = w^{(\sqrt{2\log 2})}(x)$, uniformly in x . Finally, by Lemma 17 we have

$$G_n^1(x) < G_{n,\beta}(x + m_{n,\beta}) < G_n^2(x)$$

for $x > 0$, and the opposite inequalities hold for $x < 0$. Since $w^{(\alpha)}(x) \rightarrow w^{(\sqrt{2\log 2})}(x)$ as $\alpha \uparrow \sqrt{2\log 2}$, uniformly in x , we conclude that the sequence $G_{n,\beta}(x + m_{n,\beta})$ is convergent and hence by Theorem 15 (b) that $m_{n,\beta} \equiv (G_{n,\beta})^{-1}(\frac{1}{2})$ is given by (45). Hence we have shown the existence of a bounded sequence $c(n)$ such that when the left hand side of (34) is multiplied by $c(n)$, the product converges in distribution to some non-trivial random variable Z . As the Z_n 's satisfy the 'smoothing recursion'

$$Z_{n+1} \stackrel{d}{=} e^{\beta\xi_0} Z_n^{(0)} + e^{\beta\xi_1} Z_n^{(1)},$$

it follows that after normalization Z is a fixed point of the smoothing transform equation

$$Z \stackrel{d}{=} e^{\beta\xi_0} Z^{(0)} + e^{\beta\xi_1} Z^{(1)}.$$

This has (up to a constant factor) the unique solution Y_β [27]. The proof of Theorem 5 is complete.

We end this Section by proving Lemma 9 and proving a similar result needed in the last section.

Proof of Lemma 9. Recall the partition function (3) and set

$$K_{n,\beta}(x) := \mathbb{E} e^{-e^{\beta x} Z_{\beta,n}}.$$

Thus

$$\phi_{\beta,n}(e^{\beta x}) = K_{n,\beta}(x + \alpha_\beta \log n)$$

with $\alpha_1 = 1/2$ and $\alpha_\beta = 3/2$ for $\beta > 1$. Proceeding as earlier we get

$$K_{n,\beta}(x) = \tilde{H}_{n,\beta}(x)$$

where $\tilde{H}_{n,\beta}$ solves the same recursion as $H_{n,\beta}$, but with initial condition

$$\tilde{H}_{0,\beta}(x) = \exp(-e^{\beta x}).$$

In the " G -language" this becomes

$$\tilde{G}_{0,\beta}(x) = \exp\left(-\frac{1}{2}e^{-\beta\sqrt{2\log 2}x}\right) = G_0^{(\beta\sqrt{2\log 2})}\left(x + \frac{\log 2}{\beta\sqrt{2\log 2}}\right)$$

and

$$\tilde{G}_{n,\beta}(x) = G_n^{(\beta\sqrt{2\log 2})}\left(x + \frac{\log 2}{\beta\sqrt{2\log 2}}\right).$$

Hence

$$(53) \quad \begin{aligned} \phi_{\beta,n}(e^{\beta x}) &= \left(\tilde{G}_{n,\beta}\left(-\frac{x}{\sqrt{2\log 2}} + \sqrt{2\log 2}n - \frac{\alpha_\beta}{\sqrt{2\log 2}}\log n\right) \right)^2 \\ &= \left(G_n^{(\beta\sqrt{2\log 2})}\left(-\frac{x}{\sqrt{2\log 2}} + m_n^{(\beta\sqrt{2\log 2})} + a_n\right) \right)^2 \end{aligned}$$

for some bounded sequence a_n (which depends on β).

As in Lemma 17 we get

$$G_n^{(\alpha)}\left(x + m_n^{(\alpha)}\right) < G_n^{(\alpha')}\left(x + m_n^{(\alpha')}\right)$$

if $\alpha < \alpha'$ and $x > 0$. Combining this inequality with eq. (53) allows us to get the $t \rightarrow 0$ asymptotics of the low temperature object $\phi_{\beta,n}(t)$ from the corresponding asymptotics in high temperature. Indeed, fix $\beta' < 1$. Then for x small enough and for some bounded sequences b_n, c_n for all n we get

$$\phi_{\beta,n}(e^{\beta x}) \geq \left(G_n^{(\beta'\sqrt{2\log 2})}\left(-\frac{x}{\sqrt{2\log 2}} + m_n^{(\beta'\sqrt{2\log 2})} + b_n\right) \right)^2 = \mathbb{E} e^{-c_n e^{\beta'x} \frac{Z_{\beta',n}}{\mathbb{E} Z_{\beta',n}}}$$

where $Z_{\beta',n}$ is the high temperature partition function. Using $e^{-x} \geq 1 - x$ for $x \geq 0$ we get

$$\mathbb{E} e^{-c_n e^{\beta'x} \frac{Z_{\beta',n}}{\mathbb{E} Z_{\beta',n}}} \geq 1 - c_n e^{\beta'x}$$

and therefore, by denoting $c = \sup_{n \geq 1} c_n$ we have

$$1 - \phi_{\beta,n}(t) \leq ct^{\beta'/\beta}.$$

Thus, fixing $\gamma < 1/\beta$ it holds that

$$1 - \phi_{\beta,n}(t) \leq C(\beta, \gamma)t^\gamma.$$

for some $C(\beta, \gamma) < \infty$ and $t \leq t(\beta, \gamma)$ with $t(\beta, \gamma) > 0$. Since this inequality is trivial for t bounded away from zero the claim follows. \square

Lemma 18. *For any $\beta > 1$, $\theta > 0$ and $q \in (0, \frac{1}{\beta})$,*

$$(54) \quad \mathbb{P}\left(\sum_{\sigma \in \Sigma_n} (\sqrt{n}\mu_1(I_\sigma))^\beta > n^\theta\right) \leq C(q)n^{-q\theta}.$$

Proof. The proof is almost identical to that of Lemma 9. We recall that we argued at the beginning of the previous page that $G_{n,\beta}(x + m_{n,\beta})$ converges uniformly. Moreover, we know by (50) that $1 - G_{0,\beta}(x) \sim e^{-\sqrt{2\log 2}x}$. Thus by part (b) of Theorem 15, $m_{n,\beta} = \sqrt{2\log 2}n - \frac{1}{2\sqrt{2\log 2}}\log n + \mathcal{O}(1)$. Lemma 17 then implies that for small enough x and any $\beta_1 < \beta$

$$\begin{aligned}
H_{n,\beta} \left(x + \frac{1}{2} \log n \right) &= G_{n,\beta} \left(-\frac{x}{\sqrt{2 \log 2}} + \sqrt{2 \log 2} n - \frac{1}{2\sqrt{2 \log 2}} \log n \right)^2 \\
&= G_{n,\beta} \left(-\frac{x}{\sqrt{2 \log 2}} + m_{\beta,n} + \mathcal{O}(1) \right)^2 \\
&\geq G_n^{(\beta_1, \beta)} \left(-\frac{x}{\sqrt{2 \log 2}} + m_{(\beta_1, \beta),n} + \mathcal{O}(1) \right)^2.
\end{aligned}$$

It follows from the proof of Theorem 3.4 and Lemma 5.3 of [44] that for $x \geq 0$, $1 - G_n^{(\beta_1, \beta)}(x + m_{(\beta_1, \beta),n}) \sim e^{-\frac{\beta_1}{\beta} \sqrt{2 \log 2} x}$. Thus we conclude that there exists a constant $C > 0$ so that for small enough x (say $x \leq -M$)

$$(55) \quad 1 - H_{n,\beta} \left(x + \frac{1}{2} \log n \right) \leq C e^{\frac{\beta_1}{\beta} x}.$$

We then have by Markov's inequality for any $q > 0$

$$\begin{aligned}
\mathbb{P} \left(\sum_{\sigma \in \Sigma_n} (\sqrt{n} \mu_1(I_\sigma))^\beta > n^\theta \right) &\leq n^{-q\theta} \mathbb{E} \left(\left(\sum_{\sigma \in \Sigma_n} (\sqrt{n} \mu_1(I_\sigma))^\beta \right)^q \right) \\
&= \tilde{C}(q) n^{-\theta q} \int_0^\infty t^{-1-q} \left(1 - H_{n,\beta} \left(\frac{1}{\beta} \log t + \frac{1}{2} \log n \right) \right) dt \\
&\leq \hat{C}(q) n^{-\theta q} \left(\int_0^{e^{-\beta M}} t^{-1-q} t^{\frac{\beta_1}{\beta^2}} dt + \int_{e^{-\beta M}}^\infty t^{-1-q} dt \right).
\end{aligned}$$

We see that for $q \in (0, \frac{1}{\beta})$ both of the integrals are finite (when we choose β_1 close enough to β). Thus we find our claim. \square

8. COMPLEMENT ON μ_1 -ALMOST EVERYWHERE LOCAL BEHAVIOR OF μ_1

In the subcritical case $\beta < 1$ there exist very good bounds for the almost sure fluctuations of the measure μ_β considered at μ_β -almost every $x \in [0, 1]$. Denoting by $I_n(x)$ the unique half-open dyadic interval of level n containing x , under rather general conditions on ξ it holds that almost surely for μ_β -almost every $x \in [0, 1]$

$$2^{-\alpha n} e^{-b\sqrt{n \log \log n}} \leq \mu_\beta(I_n(x)) \leq 2^{-\alpha n} e^{b\sqrt{n \log \log n}}$$

for all large n , where α and b are constants depending on β and the distribution of ξ ; see [35] for the precise statement of the result. In effect, the measure μ_β satisfies a kind of a law of the iterated logarithm.

In this section we consider the corresponding fluctuation problem in the critical case $\beta = 1$, i.e. the question of finding bounds $\psi(n)$, $\phi(n)$ such that almost surely, for μ_1 -almost every $x \in [0, 1]$ one has

$$\psi(n) \leq \mu_1(I_n(x)) \leq \phi(n)$$

for all large n . Clearly, the optimal fluctuation bounds cannot have the same form as in the subcritical case, as one would need to have $\alpha = 0$ above. Our method of obtaining bounds depends on Theorem 5 and thus we restrict to the case of a Gaussian ξ .

Theorem 19. *Suppose ξ is Gaussian. Then the following statements hold.*

- (1) *Let $f : \mathbb{N}_+ \rightarrow \mathbb{R}_+^*$ be a nonincreasing function converging to 0 at infinity. If $\liminf_{n \rightarrow \infty} \frac{\log f(n)}{-\sqrt{n \log(n)}} > \sqrt{2 \log 2}$ then almost surely,*

$$\mu_1(\{x : \mu_1(I_n(x)) \geq f(n) \text{ for infinitely many } n\}) = \mu_1([0, 1]).$$

- (2) *Let $f_\alpha(n) = \exp\left(-\sqrt{6 \log 2} \sqrt{n(\log n + \alpha \log \log n)}\right)$ for $\alpha > \frac{1}{3}$. Then almost surely,*

$$\mu_1(\{x : \mu_1(I_n(x)) \geq f_\alpha(n) \text{ for all but finitely many } n\}) = \mu_1([0, 1]).$$

- (3) *Almost surely, for all $k \in \mathbb{N}$*

$$\mu_1\left(\left\{x : \mu_1(I_n(x)) \leq n^{-k} \text{ for all but finitely many } n\right\}\right) = \mu_1([0, 1]).$$

Proof. We start with the proofs of (1) and (2) that can be achieved by the application of general moment estimates. We remark that these statements have analogues that can be proven by the same method for general ξ . Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be an ultimately nonincreasing function tending to 0 at infinity. We consider the μ_1 -measures of the sets

$$E_n^f = \{x : \mu_1(I_n(x)) \leq f(n)\}.$$

Let $(\eta_n)_{n \geq 1}$ be a sequence taking values in $(0, 1)$ and write

$$\begin{aligned} \mu_1(E_n^f) &= \int_0^1 \mathbf{1}_{\{\mu_1(I_n(x)) \leq f(n)\}} d\mu_1(x) = \sum_{\sigma \in \Sigma_n} \mu_1(I_\sigma) \mathbf{1}_{\{\mu_1(I_\sigma) \leq f(n)\}} \\ &\leq \sum_{\sigma \in \Sigma_n} \mu_1(I_\sigma) \left(\frac{f(n)}{\mu_1(I_\sigma)} \right)^{\eta_n} = \sum_{\sigma \in \Sigma_n} \mu_1(I_\sigma)^{1-\eta_n} f(n)^{\eta_n}. \end{aligned}$$

By the characterization (12) of the law of μ_1 , we have

$$\mathbb{E} \mu_1(E_n^f) \leq f(n)^{\eta_n} \sum_{\sigma \in \Sigma_n} \mathbb{E} e^{(1-\eta_n)X_\sigma} \mathbb{E} Y_1^{1-\eta_n} = f(n)^{\eta_n} 2^{n\eta_n^2} \mathbb{E} Y_1^{1-\eta_n}.$$

Theorem 8 implies the existence of a constant $C > 0$ such that $\mathbb{E} Y_1^{1-\eta_n} \leq C/\eta_n$, which gives

$$(56) \quad \mathbb{E} \mu_1(E_n^f) \leq C \exp(\eta_n^2 n \log 2 + \eta_n \log f(n) - \log \eta_n).$$

By solving for the zero of the derivative, the expression in the exponential is minimized for $\eta_n > 0$ by the choice

$$\eta_n^{\min} = \frac{-\log f(n)}{4n \log 2} + \frac{\sqrt{8n \log 2 + (\log f(n))^2}}{4n \log 2}.$$

To get more manageable expressions, we choose $\eta_n = \frac{-\log f(n)}{2n \log 2} < \eta_n^{\min}$ to get the estimate

$$\mathbb{E} \mu_1(E_n^f) \leq C \exp\left(-\frac{(\log f(n))^2}{4n \log 2} + \log n - \log(-\log f(n)) + \log(2 \log 2)\right).$$

Under the assumption of part (1) of the theorem, for some $\varepsilon > 0$ there exists a sequence $(n_k)_{k \geq 1}$ of indices such that

$$-\log f(n_k) \geq (\sqrt{2 \log 2} + \varepsilon) \sqrt{n_k \log n_k}$$

for all $k \geq 1$. Thus

$$\mathbb{E} \mu_1(E_{n_k}^f) \leq C' \exp \left(-\frac{(\sqrt{2 \log 2} + \varepsilon)^2 \log n_k}{4 \log 2} + \frac{1}{2} \log n_k - \frac{1}{2} \log \log n_k \right),$$

which shows that $\mathbb{E} \mu_1(E_{n_k}^f) \rightarrow 0$ as $k \rightarrow \infty$. We may thus extract a subsequence of (n_k) , for convenience still denoted by (n_k) , for which

$$\sum_{k \geq 1} \mathbb{E} \mu_1(E_{n_k}^f) < \infty,$$

implying that $\sum_{k \geq 1} \mu_1(E_{n_k}^f) < \infty$ almost surely. An application of the Borel–Cantelli lemma to the measure μ_1 allows us to conclude that almost surely the set

$$\{x \in [0, 1] : \mu_1(I_{n_k}(x)) \leq f(n_k) \text{ for all but finitely many } k\}$$

has μ_1 -measure 0. This implies the claim.

To prove (2), let f_α be as in the statement. For $f = f_\alpha$ our earlier choice of η_n is explicitly

$$\eta_n = \frac{\sqrt{3} \sqrt{\log n + \alpha \log \log n}}{\sqrt{2n \log 2}},$$

which we plug into (56) to get

$$\begin{aligned} \mathbb{E} \mu_1(E_n^{f_\alpha}) &\leq C \exp \left(-\frac{3}{2} (\log n + \alpha \log \log n) - \log \frac{\sqrt{3} \sqrt{\log n + \alpha \log \log n}}{\sqrt{2n \log 2}} \right) \\ &\leq C' \exp \left(-\log n - \frac{3\alpha + 1}{2} \log \log n \right). \end{aligned}$$

We see that for $\alpha > 1/3$

$$\sum_{n \geq 1} \mathbb{E} \mu_1(E_n^{f_\alpha}) < \infty,$$

which implies that almost surely $\sum_{n \geq 1} \mu_1(E_n^{f_\alpha}) < \infty$. The claim now follows from the Borel–Cantelli lemma.

The proof of part (3) requires the use of subtler properties of the cascade. We will prove by induction that for all $k \in \mathbb{N}_+$ the following property \mathcal{P}_k holds.

\mathcal{P}_k : For all $\gamma < k/2$, almost surely μ_1 -almost everywhere for n large enough, one has $\mu_1(I_n(x)) \leq n^{-\gamma}$.

Notice that by Theorem 3 this property holds for $k = 1$.

Suppose \mathcal{P}_k holds for some $k \in \mathbb{N}_+$. Fix $1/2 < \gamma < (k+1)/2$ and let $\varepsilon \in (k+1-2\gamma, k)$. For each $N \geq 1$ let

$$E_N = \{x \in [0, 1] : \forall n \geq N, \mu_1(I_n(x)) \leq n^{-(\gamma-1/2)}\}$$

and note that from the assumption that \mathcal{P}_k holds it follows that

$$\mu_1(\cup_{N \geq 1} E_N) = \mu_1([0, 1]).$$

Setting $f(n) = n^{\varepsilon/2-(k+1)/2}$ we have, for all $n \geq N$ and $\beta > 1$,

$$\begin{aligned} \sum_{\sigma \in \Sigma_n: I_\sigma \cap E_N \neq \emptyset} \mu_1(I_\sigma) \mathbf{1}_{\{\mu_1(I_\sigma) \geq f(n)\}} &\leq n^{-(\gamma-1/2)} \#\{\sigma \in \Sigma_n : \mu_1(I_\sigma) \geq f(n)\} \\ &\leq n^{-(\gamma-1/2)} \sum_{\sigma \in \Sigma_n} \mu_1(I_\sigma)^\beta f(n)^{-\beta} \\ &= n^{-(\gamma-1/2)} n^{-\beta(\varepsilon/2-(k+1)/2)} \sum_{\sigma \in \Sigma_n} \mu_1(I_\sigma)^\beta \\ &= n^{-\theta} \sum_{\sigma \in \Sigma_n} (n^{1/2} \mu_1(I_\sigma))^\beta, \end{aligned}$$

where $\theta = \gamma - \frac{1}{2} - \beta \frac{k-\varepsilon}{2}$. By our choice of ε we may choose $\beta > 1$ so that $\beta < \frac{2\gamma-1}{k-\varepsilon}$, which implies $\theta > 0$. Now recall that by Theorem 5, $c(n) \sum_{\sigma \in \Sigma_n} (n^{1/2} \mu_1(I_\sigma))^\beta$ converges in law to Y_β for some bounded sequence $c(n)$, and moreover by Lemma 18, for $q \in (0, 1/\beta)$ we have the uniform estimate $\mathbb{P}\left(\sum_{\sigma \in \Sigma_n} (n^{1/2} \mu_1(I_\sigma))^\beta > n^{\theta/2}\right) \leq C(q)n^{-\theta q/2}$. Consequently, there exists an integer $\ell > 2/\theta$ such that for the sequence $(n_j)_{j=1}^\infty = (j^\ell)_{j=1}^\infty$ we have, almost surely for all j large enough,

$$\sum_{\sigma \in \Sigma_{n_j}} (n_j^{1/2} \mu_1(I_\sigma))^\beta \leq n_j^{\theta/2}$$

and hence for j large enough

$$\mu_1(E_N \cap \{x : \mu_1(I_{j^\ell}(x)) \geq f(j^\ell)\}) \leq \sum_{\sigma \in \Sigma_{j^\ell}: I_\sigma \cap E_N \neq \emptyset} \mu_1(I_\sigma) \mathbf{1}_{\{\mu_1(I_\sigma) \geq f(j^\ell)\}} \leq j^{-\ell\theta/2}.$$

It follows that almost surely, for all $N \geq 1$,

$$\sum_{j^\ell \geq N} \mu_1(E_N \cap \{x : \mu_1(I_{j^\ell}(x)) \geq f(j^\ell)\}) < \infty.$$

By the Borel–Cantelli lemma, almost surely μ_1 -almost everywhere on E_N we have $\mu_1(I_{j^\ell}(x)) \leq f(j^\ell)$ for j large enough. But if $j^\ell < n \leq (j+1)^\ell$, we then also have

$$\mu_1(I_n(x)) \leq \mu_1(I_{j^\ell}(x)) \leq f(j^\ell) = (j^\ell)^{\frac{\varepsilon}{2}-\frac{k+1}{2}} \leq 2^{\ell \frac{k+1-\varepsilon}{2}} n^{\frac{\varepsilon}{2}-\frac{k+1}{2}},$$

and therefore we conclude that there exists a constant $C = C(\ell, k, \varepsilon) > 0$ such that almost surely μ_1 -almost everywhere on E_N we have

$$\mu_1(I_n(x)) \leq C f(n) = C n^{-\frac{k+1-\varepsilon}{2}}$$

for all n large enough. By our assumption we have $\mu_1(\cup_{N \geq 1} E_N) = \mu_1([0, 1])$, so we have shown that the desired conclusion holds for all $\gamma' < (k+1-\varepsilon)/2$. Since γ can be taken arbitrarily close to $(k+1)/2$ and hence ε arbitrarily close to 0, we are done. \square

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LAGA (UMR 7539), DÉPARTEMENT DE MATHÉMATIQUES, INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, 99 AVENUE JEAN-BAPTISTE CLÉMENT , 93430 VILLETANEUSE, FRANCE
E-mail address: `barral@math.univ-paris13.fr`

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68 , FIN-00014 UNIVERSITY OF HELSINKI, FINLAND
E-mail address: `antti.kupiainen@helsinki.fi`

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68 , FIN-00014 UNIVERSITY OF HELSINKI, FINLAND
E-mail address: `miika.nikula@helsinki.fi`

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68 , FIN-00014 UNIVERSITY OF HELSINKI, FINLAND
E-mail address: `eero.saksman@helsinki.fi`

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68 , FIN-00014 UNIVERSITY OF HELSINKI, FINLAND
E-mail address: `christian.webb@helsinki.fi`